A Thermal Analog of a Classical Center of Mass System Problem: An Undergraduate Experiment

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Abstract

A low cost heat transfer setup has been used to develop an experiment suitable for undergraduates. The heat lost by a hot liquid through a metal conductor is gained by a cold liquid. The experimental data is best explained by modeling the time behavior of the substances' temperatures in a way that's analogous to the classical center of mass problem. In this experiment, it is useful to define center of mass-like and relative coordinate-like temperatures. The behavior of these temperatures is consistent with experimental findings. The coupled heat transfer equations for the problem are easily solved within this framework.

Equipment

- CENCO Heat Transfer Set 52748-38U (www.cenconet.com)
- (optional) Agilent 34970A Data Acquisition Unit (see Editor's Note)

Introduction

In the apparatus shown in Fig. 1, liquid substances of certain mass and specific heats at given initial temperatures occupy their respective left and right styrofoam containers. The liquids are in contact with each other through a metal connector whose legs are immersed in the respective liquids. An experiment using the apparatus[1], shown in Fig.1, would consist of obtaining the temperature loss time rate of the hot substance and the temperature gain time rate of the cold substance. One would expect that the rate of heat loss of one substance is related to the rate of heat gain of the other substance. Such rate would obviously be related to the thermal conductivity of the metal.

In an undergraduate thermodynamic laboratory experiment, this conductivity can be quantitatively measured. At first sight, the apparatus suggests that perhaps Newton's Law of cooling[2] is applicable to the rate of heat loss of the hot substance, and the heat gained by the cold substance. In actuality, however, this is indeed the case for the heat loss or gain of only one of the substances, if the temperature of the other substance is held constant.

In general, the experiment is complicated by the fact that both temperatures can vary, and the system as a whole is coupled to the environment. In the discussion below, we show a simple way to explain the temperature time rate behavior assuming no heat loss to the environment.

Model

The starting point for the analysis of the temperature time rate behavior of each substance described above is to note that the rate of heat transfer is limited by the metal's ability to conduct. Thus, for the metal, assuming no heat loss to the environment, we write,

$$\frac{d}{dt}Q_B = \frac{k_m A}{L}(T_h - T_c), \qquad (2.1)$$

where Q_B is the bar's heat transfer, t is the time, k_m is the metal's thermal conductivity, A is the cross-sectional area, L is the length of the metal that is not in contact with the fluids, T_h is the temperature of the hot substance, and T_c is the temperature of the cold substance. The styrofoam containers are taken to have negligible specific heat, and since the system is insulated, the main loss of heat for the hot substance is through the bar, so that

$$\frac{d}{dt}Q_h = -\frac{d}{dt}Q_B \tag{2.2}$$

where Q_h is the hot liquid's heat transfer. The heat gained by the cold substance is similar to this with a different sign to get,

$$\frac{d}{dt}Q_c = \frac{d}{dt}Q_B, \tag{2.3}$$

Since the heat contained in the liquids is proportional to their temperature; i.e.,

$$Q_x = m_x c_x T_x, \tag{2.4}$$

where *x* can be *h* or *c* for hot or cold substance respectively, m_x is the mass of the substance, c_x is its specific heat, and T_x is its temperature. By combining the above Eqs. (2.1-2.4) we get,

$$C_h \frac{d}{dt}T_h = -K(T_h - T_c), \tag{2.5a}$$

for the hot liquid's temperature rate, and

$$C_c \frac{d}{dt}T_c = K(T_h - T_c), \tag{2.5b}$$

for the cold liquid's temperature rate. In Eqs.(2.5), we have defined,

$$K \equiv \frac{Ak_m}{L}, \tag{2.5c}$$

due to the metal's thermal conductivity contribution, and where $C_x = m_x c_x$, is the heat capacity for the x liquid. Eqs.(2.5) represent a coupled system of equations for the temperature behavior of the hot and cold substances with time. We might expect that due to the fact that the temperature difference between the substances gives rise to a relative temperature, and that since both of these temperatures should in the limit achieve some equilibrium temperature, then, as we find, the problem can be treated in a similar way to the well known center of mass system problem in classical mechanics[3]. Thus, we define the relative temperature, *T*, and center of mass temperature, T_E such that

$$\begin{pmatrix} T \\ T_E \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ \frac{C_h}{C_h + C_c} & \frac{C_c}{C_h + C_c} \end{pmatrix} \begin{pmatrix} T_h \\ T_c \end{pmatrix}, \tag{2.6}$$

where the heat capacities, or thermal masses, play the role here that the body masses play in the classical problem case. By inverting the matrix we obtain the relations,

$$T_h = T_E + \frac{C_c}{C_h + C_c}T, \tag{2.7a}$$

for the hot temperature, and

$$T_c = T_E - \frac{C_h}{C_h + C_c} T \qquad (2.7b)$$

for the cold temperature in terms of the center of thermal mass temperature, and relative temperature coordinates. If we now add Eqs. (2.5a, 2.5b) we obtain

$$C_h \frac{d}{dt} T_h + C_c \frac{d}{dt} T_c = 0 \qquad (2.8)$$

indicating that,

$$C_h T_h + C_c T_c = \text{constant.} \qquad (2.9)$$

For consistency, this constant must be obtainable from Eqs.(2.7) as follows: if we multiply Eq.(2.7a) by C_h and Eq.(2.7b) by C_c and add the results to obtain,

$$\text{constant} = (C_h + C_c) T_E.. \qquad (2.10)$$

Next we multiply Eq.(2.5a) by C_c; Eq.(2.5b) by C_h, and subtract the results to obtain,

$$C_h C_c \left(\frac{d}{dt} T_h - \frac{d}{dt} T_c\right) = -K(C_h + C_c)(T_h - T_c), \qquad (2.11a)$$

which, if we use the definition for the relative temperature of Eq.(2.6), we can find its rate of change in time as,

$$\frac{d}{dt} T = -\frac{K}{\mu} T, \qquad (2.11b)$$

where we have defined the reduced thermal mass as,

$$\frac{1}{\mu} \equiv \frac{1}{C_h} + \frac{1}{C_c} \ , \qquad (2.11c)$$

Equation (2.11b) can be easily effected for T to obtain,

$$T = T_o \exp\left(-\frac{K}{\mu} t\right) \qquad (2.12)$$

where $T_o \equiv T_{ho} - T_{co}$, with T_{xo} the initial temperature for the x liquid. From Eq.(2.12) we see that as $t \to \infty$, $T \to 0$, in agreement with what we expect; i.e., that both liquids will achieve an equal final temperature. In fact, their final achieved temperature, in this limit, can be seen from Eqs.(2.7) to be T_E. We can explicitly obtain T_E, from Eq.(2.9,10 or 2.6) evaluated at t = 0, to rewrite,

$$T_E = \mu\left(\frac{T_{ho}}{C_c} + \frac{T_{co}}{C_h}\right) \qquad (2.13)$$

This result is consistent with the equilibrium temperature that the substances would achieve were they mixed directly in one container. This indicates that the equilibrium temperature plays a similar role here to the role played by the center of mass coordinate in the two body problem of classical mechanics.

Experimental Results

In carrying out the actual experiment the apparatus employed is the heat transfer set mentioned earlier[1] whose arrangement is shown in Fig.1. We have used water as the substance. The initial masses of the water are m_h= 339.2gr, m_c= 344.1*gr*. A specific heat of 4190 $\frac{J}{kg\text{-}K}$,was used for water, so that μ = 715.72*J/K*. The temperatures of the hot and cold water were recorded versus time as shown in Table I for a typical data set. Also shown in the Table is the relative temperature *T*.

The quantity T_o, from Table I is 73.5°C. By plotting the natural logarithm of the ratio T/T_o versus the time, *t* in minutes, of the Table, we obtain the experimental data shown in Fig.2 as indicated by the filled circles. We have fitted the data with a simple linear regression analysis. The data is best fitted with the line

$$Ln(\frac{T}{T_o}) = -0.021t - 0.004 \ , \qquad (3.1)$$

with a correlation r^2=0.999. The metal contact is Aluminum, whose length not in contact with the water was determined to have the value of L=0.155m. Its cross-sectional area is A=1.6129X10^{-4}m^2. From Eqs.(2.5c), the thermal conductivity of the metal is given by

$$k_m = \frac{L\mu}{A}\left(\frac{K}{\mu}\right). \qquad (3.2)$$

From Eq. (2.12) and the fit of Eq. (3.1), the ratio K/μ is determined to have the value of 3.5X10^{-4}*sec*$^{-1}$. We thus obtain the experimental value of the metal conductivity as k_m=240.73$\frac{W}{m\text{-}K}$. A theoretical value of 238.50$\frac{W}{m\text{-}K}$ is inferred from available standards in the 300-400K range [4]. This yields the experimental error of 0.93% for the thermal conductivity of Aluminum in this experiment.

Conclusion

A low cost heat transfer apparatus has been employed to investigate the time rate of change of the temperature of a hot and a cold substance in contact with each other through a metal. We have identified a simple way of analyzing the experiment through the use of relative and center of thermal mass temperatures. The theoretical approach is reminiscent of the well known classical center of mass two body problem, thus rendering the exercise a very instructive one. A linear regression analysis of the experiment has been employed to determine the thermal conductivity of Aluminum. The results demonstrate that this experiment has a very low error, thus making it suitable for an undergraduate laboratory.

Acknowledgement
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References

[1] Heat transfer set from CENCO, part No. 52748-38U, \$21.50.

[2] "Mathematical Methods In The Physical Sciences", Mary. L. Boas, (John Wiley, NY, 1983),2nd Ed., pp. 346.

[3] "Mechanics", K. Symon, (Addison Wesley, MA), 3rd Ed., pp. 185.

[4] "Handbook of Chemistry and Physics", (CRC Press, 1990),

Table I

Data values obtained in the experiment. Here t is time, T_h is the hot water temperature, T_c is the cold water temperature, and T is the relative temperature between the two substance.

$T(min)$	$T_h(^\circ C)$	$T_c(^\circ C)$	$T(^\circ C)$
0	95.0	21.5	73.5
1	94.5	22.5	72.0
2	93.7	23.3	70.4
3	93.0	24.0	69.0
4	92.5	24.7	67.8
5	91.3	25.3	66.0
6	90.5	26.0	64.5
7	89.5	26.7	62.8
8	88.7	27.3	61.4
9	88.0	28.0	60.0
10	87.5	28.5	59.0
11	87.0	29.0	58.0
12	86.3	29.7	56.6
13	85.8	30.3	55.5
14	85.5	30.8	54.7
15	85.3	31.5	53.8
16	84.5	32.0	52.5
17	83.7	32.5	51.2
18	83.0	33.0	50.0
19	82.0	33.3	48.7
20	81.5	33.6	47.9
25	79.3	35.5	43.8
30	76.5	37.5	39.0

Fig. 1

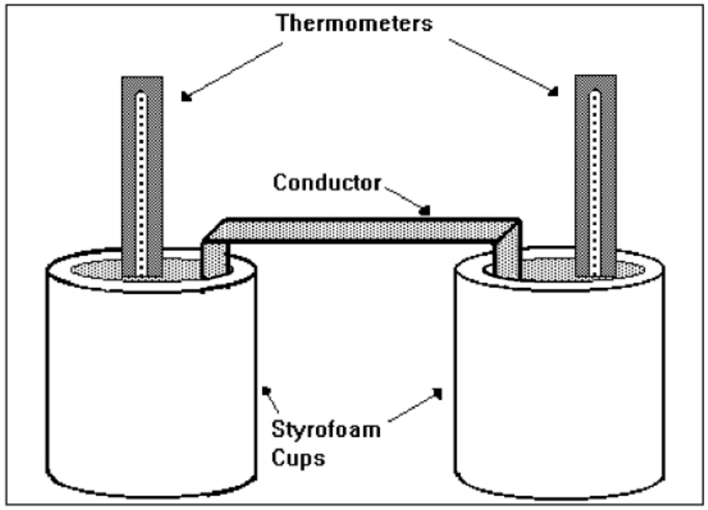

Fig.1
A schematic representation of the thermal heat transfer set used in the experiment. The actual apparatus used is "Heat transfer set from CENCO, part No. 52748-38U, $21.50".

Fig.2

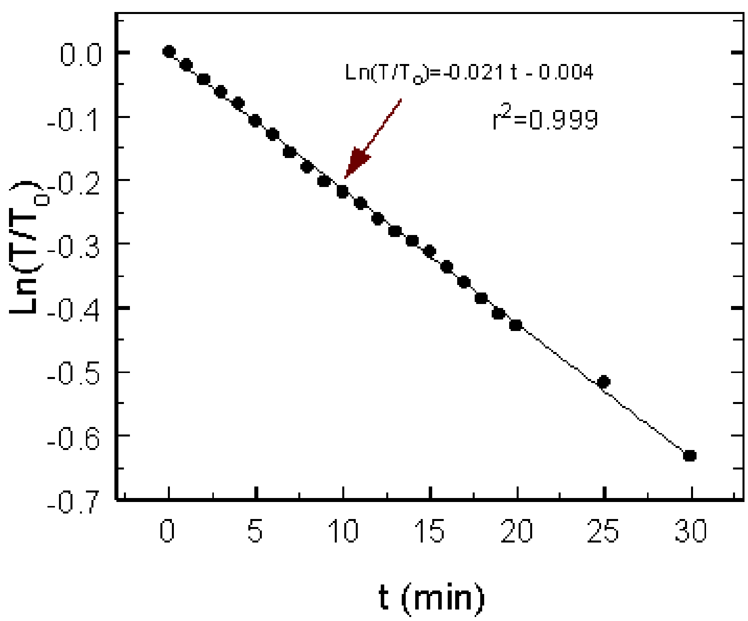

Fig.2
A plot of the data in the form of $Ln(T/T_o)$ versus time in minutes is indicated by the solid circles. The linear fit to the data is indicated by the solid line.

Editor's Notes

The experiment as written requires no electronic measuring equipment; however, it would be a great candidate for an Agilent 34970 Data Acquisition Unit.

A single Agilent 34970A could record the results of a single student setup, or simultaneously record the termperatures on many setups. Data is displayed by the PC software that comes with the Agilent 34970A and the data set can be sent to another software package, such as Excel or Mathcad, for further analysis.

The automated method has the advantages of:

- Getting students familiar with the way temperature data is usually measured in industry
- Allowing optional overnight data acquisition, thus giving more complete characterization of the experiment.
- Seeing a real-time stripchart plot of the temperatures.